On irreducible partials of Ricci tensor traceless part in finite space-time region in GR

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Abstract

Riemann tensor irreducible part $E_{iklm} = \frac{1}{2}(g_{il}S_{km} + g_{km}S_{il} - g_{im}S_{kl} - g_{kl}S_{im})$ constructed from metric tensor g_{ik} and traceless part of Ricci tensor $S_{ik} = R_{ik} - \frac{1}{4}g_{ik}R$ is expanded into bilinear combinations of bivectorial fields being eigenfunctions of E. Field equations for the bivectors induced by Bianchi identities are studied and it is shown that in general case it will be 3-parametric local symmetry group Yang-Mills field.

1 Introduction

It is well known that Einstein equations in General Relativity join together pure geometrical quantities in the left side with physical quantinies (energy-momentum tensor of matter) in the right.

But this fact means that geometry put very rigid restrictions on energy-momentum tensor and therefore on configurations of all physical fields. Any permitted mode of physical field has correspondent eigen-mode of gravitational field otherwise this mode should be prohibited.

We may study geometry types using curvature classifications. There are two types of curvature classifications: classification of Ricci tensor by J. Plebansky [2] and Petrov classification of Weyl tensor [3]. Both based on studying of eigenvectors of some tensors in given point of space-time. But eigenvectors of Ricci tensor have not an immediate physical sense and Weyl tensor types say a little about sources of gravitational field because it is not affected on Einstein equations.

On the other hand Rainich-Misner-Wheeler already unified theory of electromagnetic field [1, 4] is not a classification at all. However it allow to represent curvature of very restricted class of space-times as a construction of field quantities in finite region of space-time. Meaning of Rainich conditions is discussed in the second section.

Next section is dedicated to eigenbivectors of irreducible part E_{iklm} of Riemann tensor and its differenial properties. Such approach allows to generalize already unified theory for sourceless SU(2) Yang-Mills field in the fourth section.

In the last section the general case of gravitational field sources is discussed. It is shown that it should be 3-parameric local symmetry group (maybe non-compact or degenerated) Yang-Mills field with or without sources.

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There are five appendicies: on bivectors, on curvature properties, on electromagnetic energy-momentum tensor structure, on exictance of conformal transformation provided vanishing of the scalar curvature, and details of expansive calculations.

In all tensor expressions latin indices run over (0,1,2,3), greek indices - (1,2,3). Semicolon means covariant deriviation.

2 Rainich conditions

If curvature satisfies following conditions

$$R^i_m R^m_k = \frac{1}{4}\delta^i_k R_{mn} R^{mn}, \tag{2.1}$$

$$R = 0, \tag{2.2}$$

known as Rainich conditions then it is possible to express irreducible part of Riemann tensor E_{iklm} defined by equation (B.3) in following form

$$E_{iklm} = \frac{1}{2}(f_{ik} f_{lm} + \tilde{f}_{ik}\tilde{f}_{lm}), \tag{2.3}$$

where f_{ik} is bivector and $\tilde{f}_{ik}$ is its dual (see Appendix A) which satisfy sourceless Maxwell equations $f^{ik}_{;k} = 0, \tilde{f}^{ik}_{;k} = 0$.

Contraction of (2.3) gives

$$S_{ik} = \frac{1}{2}(f_{in} f^n_k + \tilde{f}_{in}\tilde{f}^n_k) \tag{2.4}$$

which is identical with Einstein equation. Really counting (2.2) there is Einstein tensor in the left side and energy-momentum tensor of electromagnetic field in the right. So we have self-consistent system of electromagnetic and gravitational field.

It is easy to show (see Appendix C) that Rainich conditions (2.1) and conditions of equality rank of matrix $\mathfrak{S}$ to 1 are the same.

In next section general case of matrix $\mathfrak{S}$ will be studied.

3 Eigenbivectors of E_{iklm}

Matrices A and S from (B.1) are constructed from vierbein components of Ricci tensor traceless part S_{ab}

$$S = \begin{pmatrix} S_{11} - S_{00} & S_{12} & S_{13} \\ S_{12} & S_{22} - S_{00} & S_{23} \\ S_{12} & S_{23} & S_{33} - S_{00} \end{pmatrix}, \tag{3.1}$$

$$A = \begin{pmatrix} 0 & S_{03} & -S_{02} \\ -S_{03} & 0 & S_{01} \\ S_{02} & -S_{01} & 0 \end{pmatrix}. \tag{3.2}$$

Let us define $\mathfrak{S} = S - iA$ - hermitian matrix.

Eigenvectors $\mathfrak{F}$ of matrix $\mathfrak{S}$ satisfy equations

$$
\begin{aligned}
\mathfrak{S}\mathfrak{F} &= \lambda\mathfrak{F} \\
E_{iklm} f_P^{lm} &= \lambda f_{ik}
\end{aligned}
$$

Hermitian matrix always have real eigenvalues and it is possible to express matrix $\mathfrak{S}$ through it eigenvectors

$$
\mathfrak{S}_{\alpha\beta} = \sum_{\iota=1}^{3} \epsilon_\iota \underset{\iota}{\tilde{\mathfrak{F}}}_{\alpha} \underset{\iota}{\mathfrak{F}}_{\beta}, \tag{3.3}
$$

$$
E_{iklm} = \sum_{\iota=1}^{3} \frac{\epsilon_\iota}{2} (\underset{\iota}{f}_{ik} \underset{\iota}{f}_{lm} + \underset{\iota}{\tilde{f}}_{ik} \underset{\iota}{\tilde{f}}_{lm}), \tag{3.4}
$$

$$
S_{ik} = \sum_{\iota=1}^{3} \frac{\epsilon_\iota}{2} (\underset{\iota}{f}_{ia} \underset{\iota}{f}^{a}_{k} + \underset{\iota}{\tilde{f}}_{ia} \underset{\iota}{\tilde{f}}^{a}_{k}), \tag{3.5}
$$

where $\epsilon_\iota = sign(\lambda_\iota) = \begin{cases} -1 & , \lambda_\iota < 0 \\ 0 & , \lambda_\iota = 0 \\ 1 & , \lambda_\iota > 0 \end{cases}$.

S_{ik} looks like energy-momentum tensor of Yang-Mills field with 3 parametric local symmetry group, if the group is compact and nondegenerated then it is SU(2) or O(3) group.

If scalar curvature R is zero, or if R is nonzero but we applied conformal transformation described in Appendix D then Bianchi identities (B.6,B.7) give

$$
\begin{aligned}
S^{ik}_{;k} &= 0 \qquad (3.6) \\
C^{;m}_{iklm} &= E^{;m}_{iklm} = \frac{1}{2}(S_{kl;n} - S_{kn;l}). \qquad (3.7)
\end{aligned}
$$

Second equation is consequence of first one, so it is enough to use first equation.

After substitution S_{ik} from (3.5)

$$
\sum_{\iota=1}^{3} \epsilon_\iota (\underset{\iota}{f}_{ia} \underset{\iota}{f}^{ak}_{;k} + \underset{\iota}{\tilde{f}}_{ia} \underset{\iota}{\tilde{f}}^{ak}) = 0; \tag{3.8}
$$

More general expression for divergence $\underset{\iota}{f}^{ik}_{;k}$ satisfied equation (3.8) is

$$
\begin{aligned}
\underset{1}{f}{}^{ik}_{;k} &= -\epsilon_1 \underset{1}{\tilde{f}}{}^{ik} \underset{1}{\xi}{}_k - \epsilon_2 \underset{2}{\tilde{f}}{}^{ik} \underset{3}{B}{}_k - \epsilon_3 \underset{3}{\tilde{f}}{}^{ik} \underset{2}{B}{}_k + \epsilon_2 \underset{2}{f}{}^{ik} \underset{3}{A}{}_k - \epsilon_3 \underset{3}{f}{}^{ik} \underset{2}{A}{}_k \qquad (3.9) \\
\underset{2}{f}{}^{ik}_{;k} &= -\epsilon_2 \underset{2}{\tilde{f}}{}^{ik} \underset{2}{\xi}{}_k - \epsilon_3 \underset{3}{\tilde{f}}{}^{ik} \underset{1}{B}{}_k - \epsilon_1 \underset{1}{\tilde{f}}{}^{ik} \underset{3}{B}{}_k + \epsilon_3 \underset{3}{f}{}^{ik} \underset{1}{A}{}_k - \epsilon_1 \underset{1}{f}{}^{ik} \underset{3}{A}{}_k \qquad (3.10) \\
\underset{3}{f}{}^{ik}_{;k} &= -\epsilon_3 \underset{3}{\tilde{f}}{}^{ik} \underset{3}{\xi}{}_k - \epsilon_1 \underset{1}{\tilde{f}}{}^{ik} \underset{2}{B}{}_k - \epsilon_2 \underset{2}{\tilde{f}}{}^{ik} \underset{1}{B}{}_k + \epsilon_1 \underset{1}{f}{}^{ik} \underset{2}{A}{}_k - \epsilon_2 \underset{2}{f}{}^{ik} \underset{1}{A}{}_k \qquad (3.11) \\
\underset{1}{\tilde{f}}{}^{ik}_{;k} &= +\epsilon_1 \underset{1}{f}{}^{ik} \underset{1}{\xi}{}_k + \epsilon_2 \underset{2}{f}{}^{ik} \underset{3}{B}{}_k + \epsilon_3 \underset{3}{f}{}^{ik} \underset{2}{B}{}_k + \epsilon_2 \underset{2}{\tilde{f}}{}^{ik} \underset{3}{A}{}_k - \epsilon_3 \underset{3}{\tilde{f}}{}^{ik} \underset{2}{A}{}_k \qquad (3.12) \\
\underset{2}{\tilde{f}}{}^{ik}_{;k} &= +\epsilon_2 \underset{2}{f}{}^{ik} \underset{2}{\xi}{}_k + \epsilon_3 \underset{3}{f}{}^{ik} \underset{1}{B}{}_k + \epsilon_1 \underset{1}{f}{}^{ik} \underset{3}{B}{}_k + \epsilon_3 \underset{3}{\tilde{f}}{}^{ik} \underset{1}{A}{}_k - \epsilon_1 \underset{1}{\tilde{f}}{}^{ik} \underset{3}{A}{}_k \qquad (3.13) \\
\underset{3}{\tilde{f}}{}^{ik}_{;k} &= +\epsilon_3 \underset{3}{f}{}^{ik} \underset{3}{\xi}{}_k + \epsilon_1 \underset{1}{f}{}^{ik} \underset{2}{B}{}_k + \epsilon_2 \underset{2}{f}{}^{ik} \underset{1}{B}{}_k + \epsilon_1 \underset{1}{\tilde{f}}{}^{ik} \underset{2}{A}{}_k - \epsilon_2 \underset{2}{\tilde{f}}{}^{ik} \underset{1}{A}{}_k \qquad (3.14)
\end{aligned}
$$

Quantities A_k looks like Yang-Mills potentials, but dependence of f_{ik} upon A_k is unknown, so they are simply vectorial coefficients.

4 Already Unified Theory of SU(2) Yang-Mills field

Let $\epsilon_\iota = 1$, $\xi_k = 0$, $B_k = 0$ then second divergence of bivectors f^{ik} gives

$$
\begin{aligned}
\underset{2}{f}{}^{ik}(\underset{3}{A}{}_{k;i} + \underset{1}{A}{}_i \underset{2}{A}{}_k) &= \underset{3}{f}{}^{ik}(\underset{2}{A}{}_{k;i} - \underset{1}{A}{}_i \underset{3}{A}{}_k) \\
\underset{3}{f}{}^{ik}(\underset{1}{A}{}_{k;i} + \underset{2}{A}{}_i \underset{3}{A}{}_k) &= \underset{3}{f}{}^{ik}(\underset{2}{A}{}_{k;i} - \underset{2}{A}{}_i \underset{1}{A}{}_k) \\
\underset{1}{f}{}^{ik}(\underset{2}{A}{}_{k;i} + \underset{3}{A}{}_i \underset{1}{A}{}_k) &= \underset{2}{f}{}^{ik}(\underset{1}{A}{}_{k;i} - \underset{3}{A}{}_i \underset{2}{A}{}_k)
\end{aligned}
$$

Interpreting these expressions as identities and using antisymmetry of f_{ik} we obtain usual definitions of SU(2) Yang-Mills field tensors:

$$f_{ik} = A_{k;i} - A_{i;k} + [A_i, A_k].$$

Then system of equations (3.9) becomes

$$f^{ik}_{;k} + [A_k, f^{ik}] = 0$$

- sourceless SU(2) Yang-Mills field equations [5].

Einstein equations are already satisfied.

5 Field equations in general case

Now we returning to general case of eigenbivectors. All expansive calculations are moved into Appendix E.

Second divergence of (3.9-3.14) gives (E.16-E.18). It is not so easy to express eigenbivectors f_{ik} through their potentials like in previous section.

Expressions (E.16-E.18) as well as bivectors Ξ (E.13-E.15) are invariants of gauge group of dual rotation (E.19-E.21).

It is possible to fix gauge requiring (E.26). Such way of gauge fixing defining 3 new scalar fields ϕ_ι

$$\phi_1 + \phi_2 + \phi_3 = 0.$$

In this gauge (E.16-E.18) take a form (E.27-E.29). Now interpreting these equations as identities we obtain expressions for eigenbivectors. They are consistent only when (E.33-E.38) are true.

Let define

$$F_{ik} = A_{k;i} - A_{i;k} + [A_i, A_k].$$

Then first 3 equations of system (3.9-3.14) take a form

$$F^{ik}_{;k} + [A_k, F^{ik}] = J^i \tag{5.1}$$

of 3-parametric group Yang-Mills field equations.

The last 3 equations of system (3.9-3.14) take a form

$$\tilde{F}^{ik}_{;k} + [A_k, \tilde{F}^{ik}] = K^i = 0 \tag{5.2}$$

these equations with consistency conditions (E.33-E.38) we interpret as field equations for sources of Yang-Mills field.

Here vectors J^k and K^k are sums of all terms (3.9-3.14) not included into (5.1,5.2) with opposite sign.

6 Conclusions

It is shown that GR Einstein equations allow as a source of the gravitational field nothing but Yang-Mills field with 3-parametric symmetry group with or without sources. This means that any other sets of fields must mimic to demonstrate same behaviour and energy-momentum tensor as eigen-modes of gravitational field otherwise them will be prohibited.

Nature and properties of sources of Yang-Mills field require more detailed and careful researches.

APPENDICES

A Bivectors and its vierbein components

Orthgonal vierbein h_i^a is defined by following expressions:

$$h_{ia}h_k^a = g_{ik}; \quad h_a^i h_{ib} = \eta_{ab} = diag\left(\; 1, \quad -1, \quad -1, \quad -1, \; \right), \tag{A.1}$$

where g_{ik} is metrical tensor.

Bivector is an antisymmetric tensor $f_{ik} = -f_{ki}$. Vierbien components of bivector $f_{ab} = h_a^i h_b^k f_{ik}$

$$f_{ab} = \begin{pmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & -h_3 & h_2 \\ -e_2 & h_3 & 0 & -h_1 \\ -e_3 & -h_2 & h_1 & 0 \end{pmatrix}.$$

Using usual remapping of bivector indices

A	1	2	3	4	5	6
ik	01	02	03	32	13	21

it is possible to write same bivecor as real 6-vector or as complex 3-vector $F = (e_1, e_2, e_3, h_1, h_2, h_3)$, $\mathfrak{F} = (e_1 + ih_1, e_2 + ih_2, e_3 + ih_3)$.

Dual bivector defined as

$$\tilde{f}_{ik} \equiv \frac{\sqrt{-g}}{2}\epsilon_{iklm}f^{lm},$$

where g is determinant of metrical tensor g_{ik} and ϵ_{iklm} is absolutely antisymmetric Levi-Civita pseudotensor, has components

$$\tilde{f}_{ab} = \begin{pmatrix} 0 & -h_1 & -h_2 & -h_3 \\ h_1 & 0 & -e_3 & e_2 \\ h_2 & e_3 & 0 & -e_1 \\ h_3 & -e_2 & e_1 & 0 \end{pmatrix}$$

$\tilde{F} = (-h_1, -h_2, -h_3, e_1, e_2, e_3)$, $\tilde{\mathfrak{F}} = (-h_1 + ie_1, -h_2 + ie_2, -h_3 + ie_3)$.

Useful identity for bivectors X_{ik} and Y_{lm}

$$X_{ia}Y_k{}^a - \tilde{X}_{ka}\tilde{Y}_i{}^a = \frac{1}{2}g_{ik}X_{ab}Y^{ab}.$$

It is possible to define so-called dual rotations with parameter φ

$$f_{ik} \to f_{ik}\cos\varphi - \tilde{f}_{ik}\sin\varphi,$$
$$\tilde{f}_{ik} \to f_{ik}\sin\varphi + \tilde{f}_{ik}\cos\varphi.$$

Vierbein components of parity conjugated contravariant bivector are the same as covariant vierbein components of original one:

$$
Pf^{ab} = f_P^{ab} = \begin{pmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & -h_3 & h_2 \\ -e_2 & h_3 & 0 & -h_1 \\ -e_3 & -h_2 & h_1 & 0 \end{pmatrix}.
$$

Contraction of any selfdual bivector $f_{ik}^{(+)} \equiv f_{ik} - i\tilde{f}_{ik}$ with any antiselfdual bivector $g_{ik}^{(-)} \equiv g_{ik} + i\tilde{g}_{ik}$ is zero $f_{ik}^{(+)} g^{(-)ik} = 0$.

B Curvature tensor and its properties

Riemann tensor defined as

$$
R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl},
$$

where $\Gamma^i_{nl} = \frac{1}{2} g^{ij} \left(\frac{\partial g_{kj}}{x^l} + \frac{\partial g_{jl}}{x^k} - \frac{\partial g_{kl}}{x^j} \right)$ is a Christoffel symbol of the second kind.

B.1 Algebraic properties

Riemann tensor has following symmetries:

$$
\begin{aligned}
R_{iklm} &= -R_{kilm} = -R_{ikml} \\
R_{iklm} &= R_{lmik} \\
R_{iklm} &+ R_{imkl} + R_{ilmk} = 0,
\end{aligned}
$$

so it has 20 indepenent components.

Contractions of Riemann tensor are known as Ricci tensor and scalar curvature:

$$
R_{ik} = R^l_{ilk}, \quad R_{ik} = R_{ki}
$$

$$
R = R^i_i
$$

Using bivectorial remapping of first and second indices pairs of Riemann tensor it is possible to rewrite it as symmetric 6x6 matrix

$$
R_{iklm} \to R_{AB} = R_{BA} = \begin{pmatrix} M & N \\ N & -M \end{pmatrix} + \begin{pmatrix} S & A \\ -A & S \end{pmatrix}, \tag{B.1}
$$

where M, N, S, A - 3x3 matrices and

$$
M_{\alpha\beta} = M_{\beta\alpha}, \; N_{\alpha\beta} = N_{\beta\alpha}, \; S_{\alpha\beta} = S_{\beta\alpha}, \; A_{\alpha\beta} = -A_{\beta\alpha},
$$

$A, B = 1..6$; $\alpha, \beta = 1..3$.

$$
M_{11} + M_{22} + M_{33} = \frac{R}{2}; \quad N_{11} + N_{22} + N_{33} = 0;
$$

Riemann tensor is expandible into following irreducible parts

$$R_{iklm} = C_{iklm} + E_{iklm} + G_{iklm}, \tag{B.2}$$

where C_{iklm} is so-called conformaly invariant Weyl tensor and

$$E_{iklm} = \frac{1}{2}(g_{il}S_{km} + g_{km}S_{il} - g_{im}S_{kl} - g_{kl}S_{im}); \tag{B.3}$$

$$G_{iklm} = \frac{R}{12}(g_{il}g_{km} - g_{im}g_{kl}); \tag{B.4}$$

$S_{ik} \equiv R_{ik} - \frac{R}{4}g_{ik}$ - Ricci tensor traceless part.

Matrices M and N of B.1 are constructed from components of Weyl tensor C_{iklm} and scalar curvature R and matrices A and S - from components of E_{iklm} (or S_{ik}).

B.2 Differential properties

Riemann tensor satisfies Bianchi identities

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0, \tag{B.5}$$

and contracted Bianchi identities

$$R^m_{ikl;m} + R_{ik;l} - R_{il;k} = 0, \tag{B.6}$$

$$(R^i_k - \frac{1}{2}R\delta^i_k)_{;i} = 0. \tag{B.7}$$

C Structure of electromagnetic field energy-momentum tensor

Energy-momentum tensor of electromagnetic field is defined by following expression:

$$T_{ik} = -f_{ia}f^a_k + \frac{1}{4}g_{ik}f_{ab}f^{ab} = -\frac{1}{2}(f_{ia}f^a_k + \widetilde{f_{ia}}\widetilde{f^a_k}).$$

It is possible to express its vierbein components through electromagnetic field components either in real bivector form or in complex 3-dimensional vector $\mathfrak{F} = (e_1+ih_1, e_2+ih_2, e_3+ih_3)$ and complex conjugated vector $\bar{\mathfrak{F}} = (e_1-ih_1, e_2-ih_2, e_3-ih_3)$ following way:

$$
\begin{array}{lcl}
T_{00} = & \frac{1}{2}(e_1^2+e_2^2+e_3^2+h_1^2+h_2^2+h_3^2) & = \dfrac{1}{2}(\bar{\mathfrak{F}}_1\mathfrak{F}_1+\bar{\mathfrak{F}}_2\mathfrak{F}_2+\bar{\mathfrak{F}}_3\mathfrak{F}_3), \\
T_{11} = & \frac{1}{2}(-e_1^2+e_2^2+e_3^2-h_1^2+h_2^2+h_3^2) & = \dfrac{1}{2}(-\bar{\mathfrak{F}}_1\mathfrak{F}_1+\bar{\mathfrak{F}}_2\mathfrak{F}_2+\bar{\mathfrak{F}}_3\mathfrak{F}_3), \\
T_{22} = & \frac{1}{2}(e_1^2-e_2^2+e_3^2+h_1^2-h_2^2+h_3^2) & = \dfrac{1}{2}(\bar{\mathfrak{F}}_1\mathfrak{F}_1-\bar{\mathfrak{F}}_2\mathfrak{F}_2+\bar{\mathfrak{F}}_3\mathfrak{F}_3), \\
T_{33} = & \frac{1}{2}(e_1^2+e_2^2-e_3^2+h_1^2+h_2^2-h_3^2) & = \dfrac{1}{2}(\bar{\mathfrak{F}}_1\mathfrak{F}_1+\bar{\mathfrak{F}}_2\mathfrak{F}_2-\bar{\mathfrak{F}}_3\mathfrak{F}_3), \\
T_{01} = & -e_2h_3+h_2e_3 & = \dfrac{i}{2}(\bar{\mathfrak{F}}_2\mathfrak{F}_3-\bar{\mathfrak{F}}_3\mathfrak{F}_2), \\
T_{02} = & e_1h_3-h_1e_3 & = \dfrac{i}{2}(-\bar{\mathfrak{F}}_1\mathfrak{F}_3+\bar{\mathfrak{F}}_3\mathfrak{F}_1), \\
T_{03} = & -e_1h_2+h_1e_2 & = \dfrac{i}{2}(\bar{\mathfrak{F}}_1\mathfrak{F}_2-\bar{\mathfrak{F}}_2\mathfrak{F}_1), \\
T_{12} = & -e_1e_2-h_1h_2 & = -\dfrac{1}{2}(\bar{\mathfrak{F}}_1\mathfrak{F}_2+\bar{\mathfrak{F}}_2\mathfrak{F}_1), \\
T_{13} = & -e_1e_3-h_1h_3 & = -\dfrac{1}{2}(\bar{\mathfrak{F}}_1\mathfrak{F}_3+\bar{\mathfrak{F}}_3\mathfrak{F}_1), \\
T_{23} = & -e_2e_3-h_2h_3 & = -\dfrac{1}{2}(\bar{\mathfrak{F}}_2\mathfrak{F}_3+\bar{\mathfrak{F}}_3\mathfrak{F}_2).
\end{array}
$$

It is evident that previous formulae are expressible in 3x3 hermitian matrix form:

$$
\mathfrak{S} = \begin{pmatrix} T_{11}-T_{00} & T_{12}+iT_{03} & T_{13}-iT_{02} \\ T_{12}-iT_{03} & T_{22}-T_{00} & T_{23}+iT_{01} \\ T_{12}+iT_{02} & T_{23}-iT_{01} & T_{33}-T_{00} \end{pmatrix} = -\begin{pmatrix} \bar{\mathfrak{F}}_1\mathfrak{F}_1 & \bar{\mathfrak{F}}_1\mathfrak{F}_2 & \bar{\mathfrak{F}}_1\mathfrak{F}_3 \\ \bar{\mathfrak{F}}_2\mathfrak{F}_1 & \bar{\mathfrak{F}}_2\mathfrak{F}_2 & \bar{\mathfrak{F}}_2\mathfrak{F}_3 \\ \bar{\mathfrak{F}}_3\mathfrak{F}_1 & \bar{\mathfrak{F}}_3\mathfrak{F}_2 & \bar{\mathfrak{F}}_3\mathfrak{F}_3 \end{pmatrix}.
$$

Matrix $\mathfrak{S}$ has a rank 1 i.e. all its subdeterminants are zero. It is easy to prove that former statement is equivalent to so-called Rainich conditions [1, 4]:

$$
T_{ia}T_k^a = \frac{1}{4}g_{ik}T_{ab}T^{ab}.
$$

D On existence of conformal transformation provided vanishing of the scalar curvature

Let given Riemannian space V_4 with metric g_{ik}, Riemann tensor R_{iklm}, Ricci tensor $R_{ik} = R^a_{iak}$ and scalar curvarure $R = R^a_a \not\equiv 0$. We shell find conformal transformation

$$
\begin{array}{rcl}
g_{ik} & \to & \bar{g}_{ik} = \varphi g_{ik}, \\
R_{iklm} & \to & \bar{R}_{iklm}, \\
R_{ik} & \to & \bar{R}_{ik}, \\
R & \to & \bar{R} = 0,
\end{array}
$$

which provides vanishing of $\bar{R}$. Riemann tensor of conformal metric is

$$
\begin{aligned}
\bar{R}_{iklm} = \varphi R_{iklm} \quad &+ \quad \frac{1}{2}(g_{im}\varphi_{kl} + g_{kl}\varphi_{im} - g_{il}\varphi_{km} - g_{km}\varphi_{il}) \\
&- \quad \frac{3}{4\varphi}(g_{im}\varphi_k\varphi_l + g_{kl}\varphi_i\varphi_m - g_{il}\varphi_k\varphi_m - g_{km}\varphi_i\varphi_l) \\
&+ \quad \frac{1}{4\varphi}(g_{im}g_{kl} - g_{km}g_{il})\varphi_n\varphi^n,
\end{aligned}
$$

where $\varphi_i \equiv \nabla_i\varphi \quad \varphi_{ik} \equiv \nabla_i\nabla_k\varphi$. Then

$$
\begin{aligned}
\bar{R}_{ik} &= R_{ik} - \frac{\varphi_{ik}}{\varphi} - \frac{1}{2\varphi}g_{ik}\nabla_n\nabla^n\varphi + \frac{3}{2\varphi^2}\varphi_i\varphi_k, \\
\bar{R} &= R - \frac{3}{\varphi}\nabla_n\nabla^n\varphi + \frac{3}{2\varphi^2}\varphi_n\varphi^n.
\end{aligned}
$$

Equating $\bar{R}$ to zero and making substitution $\varphi = \psi^2$ we obtain so-called conformal scalar field equation [6]:

$$
\nabla_i\nabla^i\psi - \frac{1}{6}R\psi = 0.
$$

E Detailed calculations

Let us introduce complex field variables to reduce expressions.

$$
\underset{\iota}{\mathfrak{A}}_{i} = \underset{\iota}{A}_{i} + i\underset{\iota}{B}_{i} \tag{E.1}
$$

$$
\underset{\iota}{\mathfrak{F}}_{ik} = \underset{\iota}{f}_{ik} + i\underset{\iota}{\tilde{f}}_{ik} \tag{E.2}
$$

$$
\underset{\iota}{\mathfrak{H}}_{ik} = \underset{\iota}{\Phi}_{ik} + i\underset{\iota}{\Theta}_{ik} \tag{E.3}
$$

so $\tilde{\mathfrak{F}} = -i\mathfrak{F}$.

Then (3.9-3.14) becomes

$$
\underset{1}{\mathfrak{F}}^{ik}{}_{;k} = i\epsilon_1\underset{1}{\mathfrak{F}}^{ik}\underset{1}{\xi}_k + \epsilon_2\underset{2}{\mathfrak{F}}^{ik}\underset{3}{\mathfrak{A}}_k - \epsilon_3\underset{3}{\mathfrak{F}}^{ik}\underset{2}{\mathfrak{A}}^*_k \tag{E.4}
$$

$$
\underset{2}{\mathfrak{F}}^{ik}{}_{;k} = i\epsilon_2\underset{2}{\mathfrak{F}}^{ik}\underset{2}{\xi}_k + \epsilon_3\underset{3}{\mathfrak{F}}^{ik}\underset{1}{\mathfrak{A}}_k - \epsilon_1\underset{1}{\mathfrak{F}}^{ik}\underset{3}{\mathfrak{A}}^*_k \tag{E.5}
$$

$$
\underset{3}{\mathfrak{F}}^{ik}{}_{;k} = i\epsilon_3\underset{1}{\mathfrak{F}}^{ik}\underset{3}{\xi}_k + \epsilon_1\underset{1}{\mathfrak{F}}^{ik}\underset{2}{\mathfrak{A}}_k - \epsilon_2\underset{2}{\mathfrak{F}}^{ik}\underset{1}{\mathfrak{A}}^*_k \tag{E.6}
$$

where * means complex conjugation.

Let introduce complex bivectorial field $\mathfrak{H}$

$$
\underset{1}{\mathfrak{H}}^{ik} = \underset{1}{\mathfrak{A}}_{[k;i]} + \epsilon_1(\underset{2}{\mathfrak{A}}^*_{[i}\underset{3}{\mathfrak{A}}^*_{k]} - i\underset{1}{\xi}'_{[i}\underset{1}{\mathfrak{A}}_{k]}) \tag{E.7}
$$

$$
\underset{2}{\mathfrak{H}}^{ik} = \underset{2}{\mathfrak{A}}_{[k;i]} + \epsilon_2(\underset{3}{\mathfrak{A}}^*_{[i}\underset{1}{\mathfrak{A}}^*_{k]} - i\underset{2}{\xi}'_{[i}\underset{2}{\mathfrak{A}}_{k]}) \tag{E.8}
$$

$$
\underset{3}{\mathfrak{H}}^{ik} = \underset{3}{\mathfrak{A}}_{[k;i]} + \epsilon_3(\underset{1}{\mathfrak{A}}^*_{[i}\underset{2}{\mathfrak{A}}^*_{k]} - i\underset{3}{\xi}'_{[i}\underset{3}{\mathfrak{A}}_{k]}) \tag{E.9}
$$

where [ ] means alternation,

$$\epsilon_1 \underset{1}{\xi}' = \epsilon_2 \underset{2}{\xi} - \epsilon_3 \underset{3}{\xi} \tag{E.10}$$

$$\epsilon_2 \underset{2}{\xi}' = \epsilon_3 \underset{3}{\xi} - \epsilon_1 \underset{1}{\xi} \tag{E.11}$$

$$\epsilon_3 \underset{3}{\xi}' = \epsilon_1 \underset{1}{\xi} - \epsilon_2 \underset{2}{\xi} \tag{E.12}$$

And real field Ξ

$$\underset{1}{\Xi}_{ik} = \underset{1}{\xi}_{[k;i]} - 2\epsilon_2 \underset{3}{A}_{[i} \underset{3}{B}_{k]} + 2\epsilon_3 \underset{2}{A}_{[i} \underset{2}{B}_{k]} \tag{E.13}$$

$$\underset{2}{\Xi}_{ik} = \underset{2}{\xi}_{[k;i]} - 2\epsilon_3 \underset{1}{A}_{[i} \underset{1}{B}_{k]} + 2\epsilon_1 \underset{3}{A}_{[i} \underset{3}{B}_{k]} \tag{E.14}$$

$$\underset{3}{\Xi}_{ik} = \underset{3}{\xi}_{[k;i]} - 2\epsilon_1 \underset{2}{A}_{[i} \underset{2}{B}_{k]} + 2\epsilon_2 \underset{1}{A}_{[i} \underset{1}{B}_{k]} \tag{E.15}$$

Due to vanishing of second divergence of any bivector

$$\epsilon_2 \underset{2}{\mathfrak{F}}^{ik} \underset{3}{\mathfrak{H}}_{ik} - \epsilon_3 \underset{3}{\mathfrak{F}}^{ik} \underset{2}{\mathfrak{H}}^*_{ik} + i\epsilon_1 \underset{1}{\mathfrak{F}}^{ik} \underset{1}{\Xi}_{ik} = 0 \tag{E.16}$$

$$\epsilon_3 \underset{3}{\mathfrak{F}}^{ik} \underset{1}{\mathfrak{H}}_{ik} - \epsilon_1 \underset{1}{\mathfrak{F}}^{ik} \underset{3}{\mathfrak{H}}^*_{ik} + i\epsilon_2 \underset{2}{\mathfrak{F}}^{ik} \underset{2}{\Xi}_{ik} = 0 \tag{E.17}$$

$$\epsilon_1 \underset{1}{\mathfrak{F}}^{ik} \underset{2}{\mathfrak{H}}_{ik} - \epsilon_2 \underset{2}{\mathfrak{F}}^{ik} \underset{1}{\mathfrak{H}}^*_{ik} + i\epsilon_3 \underset{3}{\mathfrak{F}}^{ik} \underset{3}{\Xi}_{ik} = 0 \tag{E.18}$$

Transformations of the fields under dual rotations

$$\underset{\iota}{\mathfrak{F}} \rightarrow e^{-i\epsilon_\iota \alpha_\iota} \underset{\iota}{\mathfrak{F}} \tag{E.19}$$

$$\underset{\iota}{\mathfrak{A}} \rightarrow e^{-i\epsilon_\iota \alpha'_\iota} \underset{\iota}{\mathfrak{A}} \tag{E.20}$$

$$\underset{\iota}{\mathfrak{H}} \rightarrow e^{-i\epsilon_\iota \alpha'_\iota} \underset{\iota}{\mathfrak{H}} \tag{E.21}$$

where

$$\epsilon_1 \underset{1}{\alpha}' = \epsilon_2 \underset{2}{\alpha} - \epsilon_3 \underset{3}{\alpha} \tag{E.22}$$

$$\epsilon_2 \underset{2}{\alpha}' = \epsilon_3 \underset{3}{\alpha} - \epsilon_1 \underset{1}{\alpha} \tag{E.23}$$

$$\epsilon_3 \underset{3}{\alpha}' = \epsilon_1 \underset{1}{\alpha} - \epsilon_2 \underset{2}{\alpha} \tag{E.24}$$

Ξ is invariant under dual rotations. It is evident that equations (E.16-E.18) are also invariant.

Let

$$\frac{\varphi_2}{\varphi_3} = e^{\phi_1}, \quad \frac{\varphi_3}{\varphi_1} = e^{\phi_2}, \quad \frac{\varphi_1}{\varphi_2} = e^{\phi_3}, \tag{E.25}$$

$$\phi_1 + \phi_2 + \phi_3 = 0;$$

where φ_ι are arbitrary positive real scalar functions.
To solve (E.16-E.18) if is enough to fix gauge requiring

$$\epsilon_1\varphi_1\underset{1}{\mathfrak{F}}^{ik}\underset{1}{\Xi}_{ik} + \epsilon_2\varphi_2\underset{2}{\mathfrak{F}}^{ik}\underset{2}{\Xi}_{ik} + \epsilon_3\varphi_3\underset{3}{\mathfrak{F}}^{ik}\underset{3}{\Xi}_{ik} = 0 \tag{E.26}$$

Then

$$\epsilon_2\underset{2}{\mathfrak{F}}^{ik}(\underset{3}{\mathfrak{H}}_{ik} - ie^{-\phi_3}\underset{2}{\Xi}_{ik}) = \epsilon_3\underset{3}{\mathfrak{F}}^{ik}(\underset{2}{\mathfrak{H}}^*_{ik} + ie^{\phi_2}\underset{3}{\Xi}_{ik}) \tag{E.27}$$

$$\epsilon_3\underset{3}{\mathfrak{F}}^{ik}(\underset{1}{\mathfrak{H}}_{ik} - ie^{-\phi_1}\underset{3}{\Xi}_{ik}) = \epsilon_1\underset{1}{\mathfrak{F}}^{ik}(\underset{3}{\mathfrak{H}}^*_{ik} + ie^{\phi_3}\underset{1}{\Xi}_{ik}) \tag{E.28}$$

$$\epsilon_1\underset{1}{\mathfrak{F}}^{ik}(\underset{2}{\mathfrak{H}}_{ik} - ie^{-\phi_2}\underset{1}{\Xi}_{ik}) = \epsilon_2\underset{2}{\mathfrak{F}}^{ik}(\underset{1}{\mathfrak{H}}^*_{ik} + ie^{\phi_1}\underset{2}{\Xi}_{ik}) \tag{E.29}$$

So

$$\epsilon_1\underset{1}{\mathfrak{F}}_{ik} = \underset{1}{\mathfrak{H}}_{ik} - ie^{-\phi_1}\underset{3}{\Xi}_{ik} = \underset{1}{\mathfrak{H}}^*_{ik} + ie^{\phi_1}\underset{2}{\Xi}_{ik} \tag{E.30}$$

$$\epsilon_2\underset{2}{\mathfrak{F}}_{ik} = \underset{2}{\mathfrak{H}}_{ik} - ie^{-\phi_2}\underset{1}{\Xi}_{ik} = \underset{2}{\mathfrak{H}}^*_{ik} + ie^{\phi_2}\underset{3}{\Xi}_{ik} \tag{E.31}$$

$$\epsilon_3\underset{3}{\mathfrak{F}}_{ik} = \underset{3}{\mathfrak{H}}_{ik} - ie^{-\phi_3}\underset{2}{\Xi}_{ik} = \underset{3}{\mathfrak{H}}^*_{ik} + ie^{\phi_3}\underset{1}{\Xi}_{ik} \tag{E.32}$$

Set of consistency conditions of system (E.30-E.32) is

$$\underset{1}{\Theta}_{ik} = \tfrac{e^{\phi_1}}{2}\underset{2}{\Xi}_{ik} + \tfrac{e^{-\phi_1}}{2}\underset{3}{\Xi}_{ik} \tag{E.33}$$

$$\underset{2}{\Theta}_{ik} = \tfrac{e^{\phi_2}}{2}\underset{3}{\Xi}_{ik} + \tfrac{e^{-\phi_2}}{2}\underset{1}{\Xi}_{ik} \tag{E.34}$$

$$\underset{3}{\Theta}_{ik} = \tfrac{e^{\phi_3}}{2}\underset{1}{\Xi}_{ik} + \tfrac{e^{-\phi_3}}{2}\underset{2}{\Xi}_{ik} \tag{E.35}$$

$$\underset{1}{\tilde{\Phi}}_{ik} = \tfrac{e^{\phi_1}}{2}\underset{2}{\Xi}_{ik} - \tfrac{e^{-\phi_1}}{2}\underset{3}{\Xi}_{ik} \tag{E.36}$$

$$\underset{2}{\tilde{\Phi}}_{ik} = \tfrac{e^{\phi_2}}{2}\underset{3}{\Xi}_{ik} - \tfrac{e^{-\phi_2}}{2}\underset{1}{\Xi}_{ik} \tag{E.37}$$

$$\underset{3}{\tilde{\Phi}}_{ik} = \tfrac{e^{\phi_3}}{2}\underset{1}{\Xi}_{ik} - \tfrac{e^{-\phi_3}}{2}\underset{2}{\Xi}_{ik} \tag{E.38}$$

Now

$$\epsilon_1\underset{1}{f}_{ik} = \underset{1}{\Phi}_{ik} = \underset{1}{A}_{[k;i]} + \epsilon_1(\underset{2}{A}_{[i}\underset{3}{A}_{k]} - \underset{2}{B}_{[i}\underset{3}{B}_{k]} + \underset{1}{\xi}'_{[i}\underset{1}{B}_{k]}) \tag{E.39}$$

$$\epsilon_2\underset{2}{f}_{ik} = \underset{2}{\Phi}_{ik} = \underset{2}{A}_{[k;i]} + \epsilon_2(\underset{3}{A}_{[i}\underset{1}{A}_{k]} - \underset{3}{B}_{[i}\underset{1}{B}_{k]} + \underset{2}{\xi}'_{[i}\underset{2}{B}_{k]}) \tag{E.40}$$

$$\epsilon_3\underset{3}{f}_{ik} = \underset{3}{\Phi}_{ik} = \underset{3}{A}_{[k;i]} + \epsilon_3(\underset{1}{A}_{[i}\underset{2}{A}_{k]} - \underset{1}{B}_{[i}\underset{2}{B}_{k]} + \underset{3}{\xi}'_{[i}\underset{3}{B}_{k]}) \tag{E.41}$$

References

[1] D. Kramer et al., Exact solutions of the Einstein field equations, Berlin, 1980.

[2] J. Plebansky, The algebraic struct. of the tensor of matter, Acta Phys. Polon., **26**, 963

[3] A.Z. Petrov, Classification of spaces determining gravitation fields, Uch. zap. Kaz. gos. univer., **114**, 55, 1954 (in Russian).

[4] G.Y. Rainich, Electrodynamics in the general relativity theory, Trans. Am. Math. Soc., **27**, 106, 1925.

[5] A.A. Sokolov et al., Gauge Fields, Moscow, MSU Publ., 1986 (in Russian).

[6] A.A. Grib, S.G. Mamaev, V.M. Mostepanenko, Vacuum quantum effects in strong fields, Moscow, Energoatomizdat, 1988 (in Russian).